PERGAMON

Applied Mathematics Letters 13 (2000) 85–89

Applied Mathematics Letters

www.elsevier.nl/locate/aml

A Three-Point Boundary Value Problem with an Integral Condition for Parabolic Equations with the Bessel Operator

M. Denche and A. L. Marhoune
Institut de Mathematiques
Université Mentouri Constantine
25000 Constantine, Algeria

(Received and accepted July 1999)

Communicated by R. P. Srivastav

Abstract—In this paper, we study a three-point boundary value problem with an integral condition for a class of parabolic equation with Bessel operator. The existence and uniqueness of the solution in functional weighted Sobolev space are proved. The proof is based on two sided *a priori* estimates and the density of the range of the operator generated by the considered problem. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Parabolic equation, Three-point boundary condition, Integral boundary condition, Energy inequalities, Weighted sobolev space.

1. INTRODUCTION

In the rectangle $\Omega = (0,T) \times (0,\ell)$, we consider the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x}\right) = f(t,x). \tag{1}$$

To equation (1), we add the initial condition,

$$lu = u(0,x) = \varphi(x), \qquad x \in (0,\ell), \tag{2}$$

the Dirichlet condition,

$$u(t,\ell) = 0, \qquad t \in (0,T), \tag{3}$$

and integral condition,

$$\int_{\ell_1}^{\ell} u(t,\xi)\,d\xi = 0, \qquad 0 \le \ell_1 \le \ell, \quad t \in (0,T). \tag{4}$$

PII: S0893-9659(00)00060-4

Here, we assumed that the function φ satisfies the conditions given in (3) and (4), i.e.,

$$\varphi(\ell) = 0, \qquad \int_{\ell_1}^{\ell} \varphi(x)\, dx = 0.$$

The importance of problems with integral conditions has been pointed out by Samarskii [1]. Regular case of this problem is studied in [2]. The problem where the equation contain an operator of the form $\frac{\partial}{\partial x}(a(t,x)\frac{\partial u}{\partial x})$, instead of Bessel operator, is treated in [3]. Similar problems for second-order parabolic equations are investigated by the potential method in [4]. Two-point boundary value problems for parabolic equations, with an integral condition, are investigated using the energy inequalities method in [5–9] and the Fourier method [10].

We associate to problems (1)–(4) the operator $L = (\mathcal{L}, l)$, defined from E into F, where E is the Banach space of functions $u \in L_2(\Omega)$, satisfying (3) and (4), with the finite norm

$$\|u\|_E^2 = \int_\Omega \theta(x) \left|\frac{\partial u}{\partial t}\right|^2 dx\, dt + \int_\Omega \left|\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right)\right|^2 dx\, dt + \sup_{0 \le t \le T} \int_0^\ell \theta(x) \left|\frac{\partial u}{\partial x}\right|^2 dx, \tag{5}$$

and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi)$ obtained by completion of the space $L_2(\Omega) \times W_2^2(0,\ell)$ with respect to the norm

$$\|\mathcal{F}\|_F^2 = \|(f,\varphi)\|_F^2 = \int_\Omega \theta(x)\, |f(t,x)|^2\, dx\, dt + \int_0^\ell \theta(x) \left|\frac{d\varphi}{dx}\right|^2 dx, \tag{6}$$

where

$$\theta(x) = \begin{cases} \ell_1 x, & 0 < x \le \ell_1, \\ x^2, & \ell_1 \le x < \ell. \end{cases} \tag{7}$$

Using the energy inequalities method proposed in [9], we establish two-sided *a priori* estimates. Then we prove that the operator L is a linear homeomorphism between the spaces E and F.

2. TWO-SIDED *A PRIORI* ESTIMATES

THEOREM 1. *The following a priori estimate:*

$$\|Lu\|_F \le C\, \|u\|_E, \tag{8}$$

holds for any function $u \in E$, where the constant C is independent of u.

PROOF. Using equation (1) and initial condition (2), we obtain

$$\int_\Omega \theta(x)\, |\mathcal{L}u|^2\, dx \le 2 \int_\Omega \left[\theta(x) \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\right|^2\right] dx\, dt \tag{9}$$

and

$$\int_0^\ell \theta(x) \left|\frac{dlu}{dx}\right|^2 dx \le \sup_{0\le t \le T} \int_0^\ell \theta(x) \left|\frac{\partial u}{\partial x}\right|^2 dx, \tag{10}$$

combining inequalities (9) and (10), we obtain (8) for $u \in E$.

THEOREM 2. *For any function $u \in E$, we have the inequality*

$$\|u\|_E \le C\, \|Lu\|_F, \tag{11}$$

where the constant C is independent of u.

PROOF. We denote $D(L) = \{u \in E : x\frac{\partial^2 u}{\partial x \partial t} \in L_2(\Omega)\}$,

$$J_x g = \int_x^\ell g(t,\xi)\, d\xi,$$

and

$$Mu = \begin{cases} \ell_1 xu, & 0 < x \le \ell_1, \\ x^2u + xJ_xu, & \ell_1 \le x < \ell, \end{cases}$$

multiplying equation (1) by $M\overline{\frac{\partial u}{\partial t}}$ and integrating over Ω^τ, where $\Omega^\tau = (0,\tau) \times (0,\ell)$, we get

$$\begin{aligned} \operatorname{Re}\int_0^\tau\int_0^\ell \mathcal{L}uM\overline{\frac{\partial u}{\partial t}}\,dx\,dt &= \int_0^\tau\int_0^\ell \theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt \\ +\operatorname{Re}\int_0^\tau\int_0^\ell \theta(x)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\,dx\,dt &+ \frac{1}{2}\int_0^\tau\int_{\ell_1}^\ell \left|J_x\frac{\partial u}{\partial t}\right|^2 dx\,dt. \end{aligned} \tag{12}$$

Integrating by parts $\operatorname{Re}\int_0^\tau\int_0^\ell \theta(x)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\,dx\,dt$, we obtain

$$\begin{aligned} \operatorname{Re}\int_0^\tau\int_0^\ell \mathcal{L}uM\overline{\frac{\partial u}{\partial t}}\,dx\,dt + \frac{1}{2}\int_0^\ell \theta(x)\left|\frac{dlu}{dx}\right|^2 dx &= \int_0^\tau\int_0^\ell \theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt \\ +\frac{1}{2}\int_0^\ell \theta(x)\left|\frac{\partial u(\tau,x)}{\partial x}\right|^2 dx &+ \frac{1}{2}\int_0^\tau\int_{\ell_1}^\ell \left|J_x\frac{\partial u}{\partial t}\right|^2 dx\,dt, \end{aligned} \tag{13}$$

and using the elementary inequalities, we get

$$\begin{aligned} \operatorname{Re}\int_0^\tau\int_0^\ell \mathcal{L}uM\overline{\frac{\partial u}{\partial t}}\,dx\,dt &\le \frac{3}{2}\int_0^\tau\int_0^\ell \theta(x)\left|\mathcal{L}u\right|^2 dx\,dt \\ +\frac{1}{4}\int_0^\tau\int_0^\ell \theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt &+ \frac{1}{2}\int_0^\tau\int_{\ell_1}^\ell \left|J_x\frac{\partial u}{\partial t}\right|^2 dx\,dt. \end{aligned} \tag{14}$$

From equation (1), we have

$$\frac{1}{4}\int_0^\tau\int_0^\ell \left|\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\right|^2 dx\,dt \le \frac{1}{2}\int_0^\tau\int_0^\ell \theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt + \frac{1}{2}\int_0^\tau\int_0^\ell \theta(x)\left|\mathcal{L}u\right|^2 dx\,dt. \tag{15}$$

Combining inequalities (14) and (15), we get

$$\begin{aligned} \frac{1}{4}\int_0^\tau\int_0^\ell \theta(x)\left|\frac{\partial u}{\partial t}\right|^2 dx\,dt &+ \frac{1}{2}\int_0^\ell \theta(x)\left|\frac{\partial u(\tau,x)}{\partial x}\right|^2 dx \\ +\frac{1}{4}\int_0^\tau\int_0^\ell \left|\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\right|^2 dx\,dt &\le 2\int_\Omega \theta(x)\left|\mathcal{L}u\right|^2 dx\,dt + \frac{1}{2}\int_0^\ell \theta(x)\left|\frac{dlu}{dx}\right|^2 dx. \end{aligned} \tag{16}$$

The right-hand side of (16) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ, in the interval $[0,T]$, we obtain the desired inequality.

3. SOLVABILITY OF THE PROBLEM

From estimates (8) and (11), it follows that the operator $L : E \longmapsto F$ is continuous and its range is closed in F. Therefore, the inverse operator L^{-1} exists and continuous, from the closed subspace $R(L)$ onto E, which means, that L is a homeomorphism from E to $R(L)$ and so to have a unique solution to the problem it remains to show that $R(L) = F$.

The proof is based on the following theorem.

THEOREM 3. *Let* $D_0(L) = \{u \in D(L) \ / \ lu = 0\}$. *If*

$$\int_\Omega \theta(x)\mathcal{L}u\,\overline{\omega}\,dx\,dt = 0, \tag{17}$$

for ω *such that* $x\omega \in L_2(\Omega)$ *and all* $u \in D_0(L)$, *then* ω *vanishes almost everywhere on* Ω.

PROOF. Putting into (17) $h = \frac{\partial u}{\partial t}$; where h, $x\frac{\partial h}{\partial x}$, $\frac{\partial}{\partial x}(x\frac{\partial h}{\partial x}) \in L_2(\Omega)$, and h verifies the boundary conditions (3) and (4), we get

$$\begin{aligned}\int_\Omega \theta(x) h\overline{\omega}\,dx\,dt &= \int_\Omega \frac{1}{x}\theta(x)\frac{\partial}{\partial x}\left(x\frac{\partial h}{\partial x}\right) J_t\overline{\omega}\,dx\,dt = \int_\Omega \frac{1}{x^2}\theta(x)\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(xh)\right) J_t\overline{\omega}\,dx\,dt \\ &\quad -2\int_\Omega \frac{1}{x^2}\theta(x)\frac{\partial}{\partial x}(xh) J_t\overline{\omega}\,dx\,dt + \int_\Omega \frac{1}{x^3}\theta(x)(xh)J_t\overline{\omega}\,dx\,dt,\end{aligned} \tag{18}$$

where $J_t\omega = \int_t^T \omega(\tau, x)\,d\tau$.

The left-hand side of (18) shows that the mapping

$$xh \longmapsto \int_\Omega \frac{1}{x}\theta(x)\frac{\partial}{\partial x}\left(x\frac{\partial h}{\partial x}\right) J_t\omega\,dx\,dt$$

is a continuous linear functional. From the right-hand side of (18), there follows that this is true if the function ω has the following properties:

$$\frac{\theta(x)}{x^3}J_t\omega,\ \frac{\theta(x)}{x^2}\frac{\partial}{\partial x}(J_t\omega),\ \frac{\theta(x)}{x^2}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(J_t\omega)\right) \in L_2(\Omega) \qquad \text{and} \qquad J_t\omega|_{x=\ell} = 0,$$

In terms of the given function ω, from equality (17), we define the function

$$v(t,x) = \begin{cases} J_t\omega, & 0 < x \le \ell_1, \\ -\int_x^\ell J_t\omega(t,\xi)\frac{1}{\xi}\,d\xi + J_t\omega, & \ell_1 \le x < \ell. \end{cases}$$

Hence, it follows that $\int_x^\ell v(t,\xi)\,d\xi = -\int_x^\ell d\xi \int_\xi^\ell J_t\omega(t,\eta)\,d\eta + \int_x^\ell J_t\omega(t,\xi)\,d\xi = x\int_x^\ell J_t\omega(t,\xi)$ $(1/\xi)\,d\xi$,

$$\theta(x)J_t\omega = Mv \qquad \text{and} \qquad \int_{\ell_1}^\ell v(t,x)\,dx = 0, \tag{19}$$

replacing $h = J_t^* v$ in (18), where $J_t^* v = \int_0^t v(\tau,x)\,d\tau$, we get

$$\int_\Omega J_t^* v \frac{\partial}{\partial t}(M\overline{v})\,dx\,dt = -\int_\Omega \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}J_t^* v\right) M\overline{v}\,dx\,dt, \tag{20}$$

doing the same calculations as in Theorem 2, where replacing $\frac{\partial u}{\partial t}$ by v, and taking into account (19), we obtain

$$\int_\Omega \theta(x)\left|v\right|^2\,dx\,dt + \frac{1}{2}\int_0^T \int_{\ell_1}^{\ell} \left|J_x v\right|^2\,dx\,dt \le 0,$$

then $v = 0$, hence $\omega = 0$. This proves Theorem 3.

THEOREM 4. *The range $R(L)$ of the operator L coincides with F.*

PROOF. Since F is a Hilbert space, we have $R(L) = F$ if and only if the equality

$$\int_\Omega \theta(x)\mathcal{L}u\overline{f}\,dx\,dt + \int_0^\ell \left(\theta(x)\frac{dlu}{dx}\frac{\overline{d\varphi}}{dx} + lu\overline{\varphi}\right)\,dx = 0, \tag{21}$$

which gives that $\mathcal{F} = (f,\varphi) = 0$. In particular, if we put $lu = 0$ in (21), then we conclude by Theorem 3 that $f = 0$, so it follows from (21) $\int_0^\ell(\theta(x)\frac{dlu}{dx}\frac{d\varphi}{dx} + lu\varphi)\,dx = 0$. Since the range of the trace operator l is dense in the Hilbert space with the norm $[\int_0^\ell(\theta(x)|\frac{d\varphi}{dx}|^2 + |\varphi|^2)\,dx]^{1/2}$, hence $\varphi = 0$. Consequently, $\mathcal{F} = 0$.

REFERENCES

1. A.A. Samarskii, Some problems in differential equations theory, *Differentsial'nye Uravnenya* **16** (11), 1221–1228, (1980).
2. N.I. Ionkin, Stability of a problem in heat-condition, *Differentsial'nye Uravnenya* **13** (2), 294–304, (1977).
3. A.V. Kartynnik, Three-point boundary-value problem with an integral space-variable condition for a second-order parabolic equation, *Differ. Equations* **26** (9), 1160–1166, (1990).
4. N.I. Kamynin, A boundary value problem in the theory of the condition with nonclassical boundary condition, *Th. Vychist. Mat. Fiz.* **43** (6), 1006–1024, (1964).
5. N.E. Benuar and N.I. Yurchuk, Mixed problem with an integral condition for parabolic equations with the Bessel operator, *Differ. Equations* **27** (12), 1482–1487, (1991).
6. A. Bouziani and N.E. Benouar, Mixed problem with integral conditions for a third order parabolic equation, *Kobe J. Math.* **15**, 47–58, (1998).
7. A. Bouziani and N.E. Benouar, Problème mixte avec conditions intégrales pour une classe d'equations paraboliques, *C. R. Acad. Sci. Paris* **321** (I), 1177–1182, (1995).
8. J.R. Cannon, The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **21** (2), 155–160, (1963).
9. N.I. Yurchuk, Mixed problem with an integral equations, *Differ. Equations* **22** (12), 1457–1463, (1986).
10. N.I. Ionkin, A problem for the heat-condition equation with a two-point boundary condition, *Differentsial'nye Uravnenya* **15** (7), 1284–1295, (1979).